We recall the definition of a continuous function on an open interval.

Definition 1. *We say $f(x)$ is continuous at x_0 if it is defined for $x \approx x_0$, and given $\epsilon > 0$, $|f(x) - f(x_0)| < \epsilon$ holds for $x \approx x_0$.*

We say $f(x)$ is continuous on the open interval I if it is continuous at every point in I.

Next, we define the continuity of functions on closed intervals with positive length.

Definition 2. *We say*
$f(x)$ is right-continuous at x_0 if given $\epsilon > 0$, $|f(x) - f(x_0)| < \epsilon$ for $x \approx x_0^+$,
$f(x)$ is left-continuous at x_0 if given $\epsilon > 0$, $|f(x) - f(x_0)| < \epsilon$ for $x \approx x_0^-$,
$f(x)$ is continuous on $[a, b]$ with $a \neq b$ if it is continuous on (a, b), right-continuous at a, and left-continuous at b.

We can define the continuity on other intervals as follows:

Definition 3. *We say $f(x)$ is continuous on $(a, b]$ if it is continuous on (a, b) and left-continuous at b. We say $f(x)$ is continuous on $[a, b)$ if it is continuous on (a, b) and right-continuous at a.*

We say $f(x)$ is continuous if the domain of $f(x)$ is an interval I with positive or infinite length, and it is continuous on I.

Let us consider some examples.

Example 4. $\tan x$ *is not continuous.*

Proof. The domain of $\tan x$ is $\mathbb{R} \setminus \{(n + \frac{1}{2})\pi : n \in \mathbb{Z}\}$. Hence, it is not an interval. □

Example 5. $\sqrt{x}$ *is continuous.*

Proof. $\sqrt{x}$ is defined on $[0, +\infty)$ which is an interval with finite length.

Given $\epsilon > 0$, $|\sqrt{x} - \sqrt{0}| = \sqrt{x} < \epsilon$ holds for $x \in [0, \epsilon^2)$, namely $\sqrt{x}$ is right-continuous at 0.

Let $x_0 > 0$. Then, given $\epsilon > 0$,

$$|\sqrt{x} - \sqrt{x_0}| = \Big|\frac{x - x_0}{\sqrt{x} + \sqrt{x_0}}\Big| = \frac{|x - x_0|}{\sqrt{x_0}} < \epsilon,$$

holds for $x \in (x_0 - \epsilon\sqrt{x_0}, x_0 + \epsilon\sqrt{x_0}) \cap (0, +\infty)$. Hence, $|\sqrt{x} - \sqrt{x_0}| < \epsilon$ in the δ-neighborhood of x_0 where $\delta = \min\{x_0, \epsilon\sqrt{x_0}\}$. □

Example 6. *$\frac{1}{x^2+1}$ is continuous.*

Proof. $\frac{1}{x^2+1}$ is defined on $(-\infty, +\infty)$ which is an interval with finite length.

Next, we can derive the following inequality.

$$|f(x) - f(x_0)| = \Big|\frac{1}{1+x^2} - \frac{1}{1+x_0^2}\Big| = \Big|\frac{x_0^2 - x^2}{(1+x^2)(1+x_0^2)}\Big|$$
$$\le |x_0^2 - x^2| = |x_0 - x||x_0 + x| \le |x_0 - x|(|x_0| + |x|).$$

If $x \in (x_0 - 1, x_0 + 1)$ then $|x - x_0| \le 1$. Hence, $|x| \le |x_0| + |x - x_0| \le |x_0| + 1$. Therefore, given ϵ, for $x \in (x_0 - 1, x_0 + 1) \cap (x_0 - \frac{\epsilon}{2|x_0|+1}, x_0 + \frac{\epsilon}{2|x_0|+1})$ the following holds

$$|f(x) - f(x_0)| \le |x_0 - x|(|x_0| + |x|) < \frac{\epsilon}{2|x_0|+1}(|x_0| + |x_0| + 1) = \epsilon.$$

Hence, $|\sqrt{x} - \sqrt{x_0}| < \epsilon$ in the δ-neighborhood of x_0 where $\delta = \min\{1, \frac{\epsilon}{2|x_0|+1}\}$. □

Example 7. *e^x is continuous.*

Proof. e^x is defined on $(-\infty, +\infty)$ which is an interval with finite length.

Next, we can derive the following inequality.

$$|f(x) - f(x_0)| = |e^x - e^{x_0}| = e^{x_0}|e^{x - x_0} - 1|.$$

Hence, given $\epsilon > 0$ we have

$$\begin{aligned} & |f(x) - f(x_0)| < \epsilon \\ \iff & |e^{x-x_0} - 1| < e^{-x_0}\epsilon \\ \iff & 1 - e^{-x_0}\epsilon < e^{x-x_0} < 1 + e^{-x_0}\epsilon \\ \iff & \ln(1 - e^{-x_0}\epsilon) < x - x_0 < \ln(1 + e^{-x_0}\epsilon) \\ \iff & x_0 - \ln(1 - e^{-x_0}\epsilon)^{-1} < x < x_0 + \ln(1 + e^{-x_0}\epsilon). \end{aligned}$$

Hence, $|e^x - e^{x_0}| < \epsilon$ in the δ-neighborhood of x_0 where

$$\delta = \min\{\ln(1 - e^{-x_0}\epsilon)^{-1}, \ln(1 + e^{-x_0}\epsilon)\}.$$

□

Example 8. *Let us define a function $f(x)$ by $f(x) = x\sin(1/x)$ for $x \neq 0$ and $f(0) = 0$. Show that $f(x)$ is continuous at 0.*

Proof. Given ϵ,

$$|f(x) - f(0)| = |x\sin(1/x) - 0| \le |x||\sin(1/x)| \le |x| < \epsilon,$$

holds for $x \in (-\epsilon, \epsilon)$. Hence, $f(x)$ is continuous at 0.

□

Example 9. *Let us define a function $f(x)$ by $f(x) = \sin(1/x)$ for $x \neq 0$ and $f(0) = b$. Show that there does not exist a number b such that $f(x)$ is continuous at 0.*

Proof. Suppose that $f(x)$ is continuous at 0 for some $b = f(0)$. Then, there exists some $\delta > 0$ such that $|f(x) - b| < 1$ for $x \in (-\delta, \delta)$.

Let $a_n = \frac{1}{(2n+\frac{1}{2})\pi}$. Then,

$$f(a_n) = \sin \frac{1}{a_n} = \sin \left(2n + \frac{1}{2}\right)\pi = 1.$$

Since $\lim a_n = 0$ by Theorem 5.1, there exists large N such that $|a_N| < \delta$.

In the same manner, we let $b_n = \frac{1}{(2n+\frac{3}{2})\pi}$. Then,

$$f(b_n) = \sin \frac{1}{b_n} = \sin \left(2n + \frac{3}{2}\right)\pi = -1.$$

Since $\lim b_n = 0$ by Theorem 5.1, there exists large M such that $|b_M| < \delta$.

Therefore, we have $|1-b| = |f(a_N)-b| < 1$ and $|1+b| = |f(b_M)-b| < 1$. Hence, we have a contradiction as follows:

$$2 \leq |1-b| + |1+b| < 2.$$

Thus, $f(x)$ can not be continuous at 0 for any $b = f(0)$.

□